

On the derivative of the Minkowski question mark function $?(x)$

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Abstract

Let $x = [0; a_1, a_2, \dots]$ be the decomposition of the irrational number $x \in [0, 1]$ into regular continued fraction. Then for the derivative of the Minkowski function $?(x)$ we prove that $?'(x) = +\infty$ provided $\limsup_{t \rightarrow \infty} \frac{a_1 + \dots + a_t}{t} < \kappa_1 = \frac{2 \log \lambda_1}{\log 2} = 1.388^+$, and $?'(x) = 0$ provided $\liminf_{t \rightarrow \infty} \frac{a_1 + \dots + a_t}{t} > \kappa_2 = \frac{4L_5 - 5L_4}{L_5 - L_4} = 4.401^+$ (here $L_j = \log \left(\frac{j + \sqrt{j^2 + 4}}{2} \right) - j \cdot \frac{\log 2}{2}$). Constants κ_1, κ_2 are the best possible. Also we prove that $?'(x) = +\infty$ holds for all x with partial quotients bounded by 4.

1. The Minkowski function $?(x)$. The function $?(x)$ is defined as follows. $?(0) = 0, ?(1) = 1$, if $?(x)$ is defined for successive Farey fractions $\frac{p}{q}, \frac{p'}{q'}$ then

$$? \left(\frac{p + p'}{q + q'} \right) = \frac{1}{2} \left(? \left(\frac{p}{q} \right) + ? \left(\frac{p'}{q'} \right) \right);$$

for irrational x function $?(x)$ is defined by continuous arguments. This function firstly was considered by H. Minkowski (see. [1], p.p. 50-51) in 1904. $?(x)$ is a continuous increasing function. It has derivative almost everywhere. It satisfies Lipschitz condition [2], [3]. It is a well-known fact that the derivative $?'(x)$ can take only two values - 0 or $+\infty$. Almost everywhere we have $?'(x) = 0$. Also if irrational $x = [0; a_1, \dots, a_t, \dots]$ is represented as a regular continued fraction with natural partial quotients then

$$?(x) = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + \dots + \frac{(-1)^{n+1}}{2^{a_1+\dots+a_n-1}} + \dots$$

These and some other results one can find for example in papers [4],[5],[2].

Here we should note the connection between function $?(x)$ and Stern-Brocot sequences. We remind the reader the definition of Stern-Brocot sequences F_n , $n = 0, 1, 2, \dots$. First of all let us put $F_0 = \{0, 1\} = \left\{ \frac{0}{1}, \frac{1}{1} \right\}$. Then for the sequence F_n treated as increasing sequence of rationals $0 = x_{0,n} < x_{1,n} < \dots < x_{N(n),n} = 1$, $N(n) = 2^n$, $x_{j,n} = p_{j,n}/q_{j,n}$, $(p_{j,n}, q_{j,n}) = 1$ we define the next sequence F_{n+1} as $F_{n+1} = F_n \cup Q_{n+1}$ where Q_{n+1} is the set of the form $Q_{n+1} = \{x_{j-1,n} \oplus x_{j,n}, i = 1, \dots, N(n)\}$. Here operation \oplus means taking the mediant fraction for two rational numbers: $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$. The Minkowski question mark function $?(x)$ is the limit distribution function for Stern-Brocot sequences:

$$?(x) = \lim_{n \rightarrow \infty} \frac{\#\{\xi \in F_n : \xi \leq x\}}{2^n + 1}.$$

2. Notation and parameters. In this paper $[0; a_1, \dots, a_t, \dots]$ denotes a regular continued fraction with natural partial quotients a_t . $k_t(a_1, \dots, a_t)$ denotes continuant. For a continued fraction under consideration the convergent fraction of order t is denoted as $p_t/q_t = [0; a_1, \dots, a_t]$ (hence, $q_t = k_t(a_1, \dots, a_t)$). We need numbers

$$\lambda_j = \frac{j + \sqrt{j^2 + 4}}{2}, \quad L_j = \log \lambda_j - j \cdot \frac{\log 2}{2}.$$

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Here $j < \lambda_j < j + 1$. Note that

$$L_2 > L_3 > L_1 > L_4 > 0 > L_5 > L_6 > \dots \quad (1)$$

and

$$\frac{L_5}{L_5 - L_4} \geq \frac{1}{2}. \quad (2)$$

Also we need the values of continuants

$$k_{l,j} = k_l(\underbrace{j, \dots, j}_l), \quad k_{0,j} = 1, \quad k_{1,j} = j.$$

From recursion $k_{l+1,j} = jk_{l,j} + k_{l-1,j}$ we deduce

$$k_{l,j} = c_{1,j}\lambda_j^l + c_{2,j}(-\lambda_j)^{-l}$$

where

$$c_{1,j} + c_{2,j} = 1, \quad c_{1,j}\lambda_j - c_{2,j}(\lambda_j)^{-1} = j.$$

Hence

$$1 - \frac{j}{j^2 + 1} < c_{1,j} < 1, \quad 0 < c_{2,j} < \frac{j}{j^2 + 1}$$

and

$$k_{l,j} < \lambda_j^l. \quad (3)$$

Also we should consider the constants

$$\kappa_1 = \frac{2 \log \lambda_1}{\log 2} = 1.388^+, \quad \kappa_2 = \frac{4L_5 - 5L_4}{L_5 - L_4} = 4.401^+. \quad (4)$$

For a natural n and a n -tuple of nonnegative integer numbers (r_1, \dots, r_n) we put $t = \sum_{j=1}^n r_j$. Now we define the set

$$W_n(r_1, \dots, r_n) = \{(a_1, \dots, a_t) : \#\{i : a_i = j\} = r_j\}.$$

Let

$$\mu_n(r_1, \dots, r_n) = \max_{(a_1, \dots, a_t) \in W_n(r_1, \dots, r_n)} k_t(a_1, \dots, a_t). \quad (5)$$

For real positive ω we define

$$\Omega_{\omega,n,t} = \left\{ (r_1, \dots, r_n) : r_j \in \mathbb{N}_0, \sum_{j=1}^n (j - \omega)r_j \geq 0, \sum_{j=1}^n r_j = t \right\}.$$

Let $\omega = \kappa_2 + \eta < 5$ and $\eta \in [0, 1/2)$. It is easy to see that for any $n \geq 5$ the following inequality is valid:

$$\max_{(r_1, \dots, r_n) \in \Omega_{\kappa_2 + \eta, n, t}} \sum_{j=1}^n r_j L_j \leq (L_5 - L_4)t\eta, \quad L_5 - L_4 < 0. \quad (6)$$

We give the proof of (6) in section 5.

Also for $r_1 \geq 1$ we consider the set

$$V_n(r_1, \dots, r_n) = \{(a_1, \dots, a_t) : \#\{i : a_i = j\} = r_j, \quad a_1 = 1\}.$$

Let

$$k[r_1, \dots, r_n] = k_t(\underbrace{1, \dots, 1}_{r_1}, \underbrace{2, \dots, 2}_{r_2}, \dots, \underbrace{n, \dots, n}_{r_n}).$$

I.D. Kan in [6] proved the following statement.

Lemma 1.

$$\max_{(a_1, \dots, a_t) \in V_n(r_1, \dots, r_n)} k_t(a_1, \dots, a_t) = k[r_1, \dots, r_n].$$

We should note that Lemma 1 is a generalization of a result from [7].

To get an upper bound for $k[r_1, \dots, r_n]$ we use formula

$$k_{t+l}(a_1, \dots, a_t, b_1, \dots, b_l) = k_t(a_1, \dots, a_t)k_l(b_1, \dots, b_l) + k_{t-1}(a_1, \dots, a_{t-1})k_{l-1}(b_2, \dots, b_l). \quad (7)$$

Let r_{h_1}, \dots, r_{h_f} , $1 \leq h_1 < \dots < h_f = n$ be all *positive* numbers from the set r_1, \dots, r_n . Here $h_j \geq j$. Then from (7) and inequalities

$$k_{r_{h_{j+1}}-1, h_{j+1}} \leq k_{r_{h_{j+1}}, h_{j+1}}/h_{j+1}, \quad k[r_1, \dots, r_{h_j} - 1] \leq k[r_1, \dots, r_{h_j}]/h_j$$

we deduce the inequality

$$\begin{aligned} k[r_1, \dots, r_{h_j}, \underbrace{0, \dots, 0}_{h_{j+1}-h_j-1}, r_{h_{j+1}}] &= k[r_1, \dots, r_{h_j}]k_{r_{h_{j+1}}, h_{j+1}} + k[r_1, \dots, r_{h_j} - 1]k_{r_{h_{j+1}}-1, h_{j+1}} \leq \\ &\leq k[r_1, \dots, r_{h_j}]k_{r_{h_{j+1}}, h_{j+1}} \left(1 + \frac{1}{h_j h_{j+1}}\right). \end{aligned}$$

Now

$$k[r_1, \dots, r_n] \leq \prod_{j=1}^n k_{r_j, j} \prod_{j=1}^{f-1} \left(1 + \frac{1}{h_j h_{j+1}}\right) \leq \prod_{j=1}^n k_{r_j, j} \prod_{j=1}^{n-1} \left(1 + \frac{1}{j(j+1)}\right). \quad (8)$$

But

$$\prod_{j=1}^{n-1} \left(1 + \frac{1}{j(j+1)}\right) \leq \prod_{j=1}^{+\infty} \left(1 + \frac{1}{j(j+1)}\right) \leq e.$$

Hence from Lemma 1, inequalities (8,3) and

$$k_t(a_1, \dots, a_t) \leq k_{t+1}(1, a_1, \dots, a_t)$$

as a corollary we deduce the following upper bound for $\mu_n(r)$:

$$\mu_n(r_1, \dots, r_n) \leq \lambda_1 e \prod_{j=1}^n \lambda_j^{r_j}. \quad (9)$$

3. A result by J. Paradis, P. Viader, L. Bibiloni. In [5] the following statement is proved.

Theorem A.

1. Let for real irrational $x \in (0, 1)$ in continued fraction expansion $x = [0; a_1, \dots, a_t, \dots]$ with κ_1 from (4) the following inequality be valid:

$$\limsup_{t \rightarrow \infty} \frac{a_1 + \dots + a_t}{t} < \kappa_1.$$

Then if $?'(x)$ exists the equality $?'(x) = +\infty$ holds.

2. Let $\kappa_3 = 5.319^+$ be the root of equation $\frac{2\log(1+x)}{\log 2} - x = 0$. Let for real irrational $x \in (0, 1)$ in continued fraction expansion $x = [0; a_1, \dots, a_t, \dots]$ holds

$$\liminf_{t \rightarrow \infty} \frac{a_1 + \dots + a_t}{t} \geq \kappa_3.$$

Then if $?'(x)$ exists the equality $?'(x) = 0$ holds.

4. New results. Here we give the stronger version of the Theorem A.

Theorem 1.

1. Let for real irrational $x \in (0, 1)$ in continued fraction expansion $x = [0; a_1, \dots, a_t, \dots]$ with κ_1 from (4) the following inequality be valid:

$$\limsup_{t \rightarrow \infty} \frac{a_1 + \dots + a_t}{t} < \kappa_1.$$

Then $?'(x)$ exists and $?'(x) = +\infty$.

2. For any positive ε there exists a quadratic irrationality x such that

$$\lim_{t \rightarrow \infty} \frac{a_1 + \dots + a_t}{t} \leq \kappa_1 + \varepsilon$$

and $?'(x) = 0$.

Theorem 2.

1. Let for real irrational $x \in (0, 1)$ in continued fraction expansion $x = [0; a_1, \dots, a_t, \dots]$ with κ_2 from (4) the following inequality be valid:

$$\liminf_{t \rightarrow \infty} \frac{a_1 + \dots + a_t}{t} > \kappa_2. \quad (10)$$

Then $?'(x)$ exists and $?'(x) = 0$.

2. For any positive ε there exists a quadratic irrationality x such that

$$\lim_{t \rightarrow \infty} \frac{a_1 + \dots + a_t}{t} \geq \kappa_2 - \varepsilon$$

and $?'(x) = +\infty$.

Theorem 3. Let in the continued fraction expansion $x = [0; a_1, \dots, a_t, \dots]$ all partial quotients a_j be bounded by 4. Then $?'(x) = \infty$.

We must note that Theorem 3 is not true if we assume that all partial quotients are bounded by 5.

Corollary. The Hausdorff dimension of the set $\{x : ?'(x) = \infty\}$ is greater than the Hausdorff dimension of the set $\mathcal{F}_4 = \{x : a_j \leq 4 \forall j\}$ which is equal to 0.7889^+ .

Here the numerical value of Hausdorff dimension for \mathcal{F}_4 is taken from [8]. Some recent results on multifractal analysis of the sets associated with values of $?'(x)$ one can find in the recent paper [9].

5. The proof of formula (6). It is sufficient to prove the inequality

$$\max_{(r_1, \dots, r_n) \in \Omega_{\kappa_2 + \eta, n, 1}} \sum_{j=1}^n r_j L_j \leq (L_5 - L_4) \eta.$$

By $e_j \in \mathbb{R}^n$ we denote the vector with all but j -th coordinates equal to zero, and with j -th coordinate equal to one. The set $\Omega_{\kappa_2 + \eta, n, 1}$ is a polytope lying in the simplex $\{r_1, \dots, r_n : r_j \geq 0, r_1 + \dots + r_n = 1\}$.

The vertices of this polytope are points $e_j, 5 \leq j \leq n$ and $e_{i,j} = \frac{\omega-i}{j-i}e_j + \frac{j-\omega}{j-i}e_i, 1 \leq i \leq 4, 5 \leq j \leq n$. The linear function $\sum_{j=1}^n r_j L_j$ attain its maximum at a vertex of polytope $\Omega_{\kappa_2+\eta,n,1}$. Now we must take into account inequalities (1,2). So we have

$$\max_{(r_1, \dots, r_n) \in \Omega_{\kappa_2+\eta,n,1}} \sum_{j=1}^n r_j L_j = \max \left\{ \max_{1 \leq i \leq 4, j \geq 5} \left(\left(\frac{4L_5 - 5L_4}{L_5 - L_4} + \frac{jL_i - iL_j}{L_j - L_i} + \eta \right) \frac{L_j - L_i}{j - i} \right), L_5 \right\}.$$

But

$$\min_{1 \leq i \leq 4, j \geq 5} \frac{jL_i - iL_j}{L_j - L_i} = \frac{5L_4 - 4L_5}{L_5 - L_4} = -\kappa_2$$

and

$$\min_{1 \leq i \leq 4, j \geq 5, (i,j) \neq (4,5)} \left(\frac{4L_5 - 5L_4}{L_5 - L_4} + \frac{jL_i - iL_j}{L_j - L_i} \right) = \frac{4L_5 - 5L_4}{L_5 - L_4} + \frac{5L_1 - L_5}{L_5 - L_1} > 0.$$

Hence

$$\begin{aligned} \max \left\{ \max_{1 \leq i \leq 4, j \geq 5} \left(\left(\frac{4L_5 - 5L_4}{L_5 - L_4} + \frac{jL_i - iL_j}{L_j - L_i} + \eta \right) \frac{L_j - L_i}{j - i} \right), L_5 \right\} = \\ = \max \left\{ \eta \max_{1 \leq i \leq 4, j \geq 5} \frac{jL_i - iL_j}{L_j - L_i}, L_5 \right\} = \eta(L_5 - L_4). \end{aligned}$$

Formula (6) is proved.

6. One Lemma useful for the proofs of the existence of the derivative of the Minkowski question mark function. To prove the existence of the derivative it is convenient to use the following statement.

Lemma 2. *For irrational x and δ small in absolute value there exist natural $t = t(x, \delta)$ and $z \in [1, a_{t+2} + 1]$ such that*

$$\frac{q_t q_{t-1}}{2^{a_1 + \dots + a_{t+1} + z}} \leq \frac{?(x + \delta) - ?(x)}{\delta}. \quad (11)$$

Also there exist natural $t' = t'(x, \delta)$ and $z' \in [1, a_{t'+2} + 1]$ such that

$$\frac{?(x + \delta) - ?(x)}{\delta} \leq \frac{(z' + 1)^2 q_{t'+1}^2}{2^{a_1 + \dots + a_{t'+1} + z' - 4}} \quad (12)$$

Proof.

It is enough to prove Lemma 2 for positive δ . Define natural n such that $F_n \cap (x, x + \delta) = \emptyset$, $F_{n+1} \cap (x, x + \delta) = \xi$. Let $(x, x + \delta) \subset [\xi^0, \xi^1]$, where ξ^0, ξ^1 are two successive points from the finite set F_n . Then $\xi = \xi^0 \oplus \xi^1$. One can easily see that for some natural t will happen $\xi^0 = p_t/q_t$. At the same time rationals ξ and ξ^1 must be among convergent fractions to x or intermediate fractions to x (intermediate fraction is a fraction of the form $\frac{p_t a + p_{t-1}}{q_t a + q_{t-1}}, 1 \leq a < a_{t+1}$). In any case, ξ^1 has the denominator $\geq q_{t-1}$. Hence

$$\delta \leq \frac{1}{q_t q_{t-1}}. \quad (13)$$

Define natural z to be minimal such that either $\xi_- = \xi^0 \oplus \underbrace{\xi \oplus \dots \oplus \xi}_z \in (x, \xi)$ or $\xi_+ = \xi^1 \oplus \underbrace{\xi \oplus \dots \oplus \xi}_z \in (\xi, x + \delta)$. Then $\xi_{--} = \xi^0 \oplus \underbrace{\xi \oplus \dots \oplus \xi}_{z-1} \leq x$ and $\xi_{++} = \xi^1 \oplus \underbrace{\xi \oplus \dots \oplus \xi}_{z-1} \geq x + \delta$. As points $\xi_{--} < \xi_- < \xi < \xi_+ < \xi_{++}$ are successive points from F_{n+z+1} and $?(x)$ increases, we have

$$\frac{1}{2^{n+z+1}} \leq \min\{\xi_+ - \xi, \xi - \xi_{--}\} \leq ?(x + \delta) - ?(x) \leq ?(\xi_{++}) - ?(\xi_{--}) = \frac{4}{2^{n+z+1}}. \quad (14)$$

Consider two cases:

(i) $\xi_- \in (x, \xi)$.

(ii) $\xi_- \notin (x, \xi)$ but then $\xi_+ \in (\xi, x + \delta)$.

In the case (i) we have $\delta > \xi - \xi_-$. If in addition (case (i1)) $z = 1$ then $\xi_- = p/q, q = z_*q_t + q_{t-1} \leq q_{t+1}, 1 \leq z_* \leq a_{t+1}, \xi = (p - p_t)/(q - q_t), n + 2 = a_1 + \dots + a_t + z_* \leq a_1 + \dots + a_{t+1}$ and

$$\delta > \frac{1}{(q - q_t)q} \geq \frac{1}{(z_* + 1)^2 q_t^2}. \quad (15)$$

If $z > 1$ (case (i2)) then $\xi = p_{t+1}/q_{t+1}, \xi_{--} = p_{t+2}/q_{t+2}, z = a_{t+2} + 1, n + 1 = a_1 + \dots + a_{t+1}$ and

$$\delta > \frac{1}{(zq_{t+1} + q_t)q_{t+1}} \geq \frac{1}{(z + 1)q_{t+1}^2}. \quad (16)$$

In the case (ii) we have $z \leq a_{t+2}, \xi = p_{t+1}/q_{t+1}, n + 1 = a_1 + \dots + a_{t+1}$. Now we deduce

$$\delta > \xi_+ - \xi \geq \frac{1}{(zq_{t+1} + q^1)q_{t+1}} \geq \frac{1}{(z + 1)q_{t+1}^2} \quad (17)$$

(here $q^1 < q_{t+1}$ is the denominator of ξ^1).

From (16,17) and the equalities for $a_1 + \dots + a_{t+1}$ the cases (i2), (ii) we get

$$\delta > \frac{1}{(z + 1)q_{t+1}^2}. \quad (18)$$

In the cases (i2), (ii) we have $a_1 + \dots + a_{t+1} - 1 \leq n + 1 \leq a_1 + \dots + a_{t+1}$. Taking into account (13,14) and (18) we obtain

$$\frac{q_t q_{t-1}}{2^{a_1 + \dots + a_{t+1} + z}} \leq \frac{?(x + \delta) - ?(x)}{\delta} \leq \frac{(z + 1)q_{t+1}^2}{2^{a_1 + \dots + a_{t+1} + z - 4}}$$

and inequalities (11,12) follows with $t = t', z = z'$. We should note that the inequality (11) also is valid for the case (i1) as we have $n + 2 \leq a_1 + \dots + a_{t+1}$ and (13,14). As for the upper bound in the case (i1) it follows from (14,15) with $t' = t - 1$, and $z' = z_*$.

Lemma 2 is proved.

7. The proof of Theorem 1. The existence of the derivative and its equality to $+\infty$ in the first statement of theorem 1 follows from the lower bound of Lemma 2 as we always have $q_t q_{t-1} \gg \lambda_1^{2t}$ and from the inequality $a_1 + \dots + a_{t+1} + a_{t+2} + 1 \leq \kappa t + o(t)$ (take into account that $\kappa = \limsup_{t \rightarrow \infty} \frac{a_1 + \dots + a_t}{t} < \kappa_1$).

In order to prove statement 2 of Theorem 1 for small positive $\eta > 0$ and natural r we define $q = r^2, m = [r(\kappa_1 - 1 + \eta)] + 1 > r(\kappa_1 - 1 + \eta)$. Now we must take the quadratic irrationality

$$x_r = [0; a_1, \dots, a_t, \dots] = [0; \underbrace{1, \dots, 1}_q, \underbrace{m, \dots, m}_r].$$

Now we see that

$$\lim_{t \rightarrow \infty} \frac{a_1 + \dots + a_t}{t} = \frac{q + mr}{q + r} \rightarrow \kappa_1 + \eta, \quad r \rightarrow \infty.$$

Moreover, taking $w = \left\lceil \frac{t}{q+r} \right\rceil$ we have

$$\frac{q_{t+1}(q_{t+1} + q_{t+2})}{2^{a_1 + \dots + a_t}} \leq \frac{12m^3(k_t(a_1, \dots, a_t))^2}{2^{a_1 + \dots + a_t}} \leq \frac{12m^3 2^{2w} \lambda_1^{2wq} \lambda_m^{2wr}}{2^{w(q+rm)}} \leq \exp((- \eta r^2 + O(r \log r))w \log 2).$$

Here in the exponent the coefficient before w is negative when r is large enough. Hence the right hand side goes to zero when $t \rightarrow \infty$. It means that $?'(x_r) = 0$.

8. The proof of the statement 1 of Theorem 2. By Lemma 2 it is sufficient to prove that $\frac{q_t^2}{2^{a_1+\dots+a_t}} \rightarrow 0$, $t \rightarrow \infty$. Define n and r_1, \dots, r_n from the condition $(a_1, \dots, a_t) \in W_n(r_1, \dots, r_n)$. Then (9) leads to

$$\frac{q_t^2}{2^{a_1+\dots+a_t}} \leq \frac{(\mu_n(r_1, \dots, r_n))^2}{2^{\sum_{j=1}^n j r_j}} \ll \exp \left(2 \sum_{j=1}^n r_j L_j \right).$$

From another hand for positive η small enough we have the following situation. For all t large enough it is true that $n \geq 5$ and $(r_1, \dots, r_n) \in \Omega_{\kappa_2+\eta, n, t}$. Now we can use (6) and we obtain inequality

$$\frac{q_t^2}{2^{a_1+\dots+a_t}} \leq \exp(2(L_5 - L_4)t\eta) \rightarrow 0, \quad t \rightarrow \infty.$$

It means that $?'(x) = 0$.

9. The proof of the statement 2 of Theorem 2. Take natural numbers $p, q \in \mathbb{N}$ such that $\kappa_2 - \varepsilon < \frac{4p+5q}{p+q} < \kappa_2$. Define

$$x_{p,q} = [0; a_1, \dots, a_t, \dots] = [0; \underbrace{4, \dots, 4}_p, \underbrace{5, \dots, 5}_q].$$

Obviously,

$$\lim_{t \rightarrow \infty} \frac{a_1 + \dots + a_t}{t} = \frac{4p + 5q}{p + q}.$$

From the other hand

$$\frac{q_t q_{t-1}}{2^{a_1+\dots+a_{t+2}}} \geq \left(\frac{\lambda_4^{2p} \lambda_5^{2q}}{2^{4p+5q}} \right)^{t+o(t)} = \exp(2(pL_4 + qL_5)(t + o(t))).$$

But $\frac{4p+5q}{p+q} < \kappa_2 = \frac{4L_5-5L_4}{L_5-L_4}$ and hence $pL_4 + qL_5 > 0$. So $\frac{q_t q_{t-1}}{2^{a_1+\dots+a_{t+2}}} \rightarrow \infty$ and $?'(x_{p,q}) = \infty$.

10. The proof of Theorem 3. First of all we see that

$$\min_{a_i \in \{1,2,3,4\}, a_1+\dots+a_t=n} k_t(a_1, \dots, a_t) \geq \tag{19}$$

$$\geq \min \left\{ \min_{a_i \in \{1,4\}, a_1+\dots+a_t=n-3} k_t(a_1, \dots, a_t), \min_{a_i \in \{1,4\}, a_1+\dots+a_t=n-2} k_t(a_1, \dots, a_t), \min_{a_i \in \{1,4\}, a_1+\dots+a_t=n} k_t(a_1, \dots, a_t) \right\}.$$

In order to do this we note that for two elements a, b with other elements fixed

$$k_t(\dots, a, \dots, b, \dots) = Mab + Na + Kb + P.$$

Here positive M, N, K, P do not depend on a, b . Then if the sum $a + b = \tau$ is fixed we have

$$k_t(\dots, a, \dots, b, \dots) = Ma(\tau - a) + Na + K(\tau - a) + P = -Ma^2 + (M\tau + N - K)a - K\tau + P.$$

So for $a, b > 1$ we can say that

$$k_t(\dots, a, \dots, b, \dots) \geq \min\{k_t(\dots, a-1, \dots, b+1, \dots), k_t(\dots, a+1, \dots, b-1, \dots)\}.$$

Hence, we can replace a pair 2, 3 of partial quotients by 1, 4 and the continuant becomes smaller. Also we can replace any pair 2, 2 of partial quotients by 1, 3 and the continuant becomes smaller. Also

we can replace any pair 3, 3 of partial quotients by 2, 4 and the continuant becomes smaller. This procedure enables one to replace the set $\{(a_1, \dots, a_t) : a_i \in \{1, 2, 3, 4\}, a_1 + \dots + a_t = n\}$ in the left hand side of (19) by the set $\{(a_1, \dots, a_t) : a_i \in \{1, 2, 3, 4\}, a_1 + \dots + a_t = n, \#\{a_i = 3\} + \#\{a_i = 2\} \leq 1\}$. Now the inequality (19) follows.

From another hand as all partial quotients are bounded by 4 we have

$$k_{t_1+t_2}(a_1, \dots, a_{t_1}, a_1, \dots, a_{t_2}) \geq (1 + \varepsilon) k_{t_1}(a_1, \dots, a_{t_1}) k_{t_2}(a_1, \dots, a_{t_2}),$$

where ε is some relatively small positive real constant. Now from the last formulas and (19) it follows that it is sufficient to prove that for every large n the following inequality is valid

$$\min_{a_1 + \dots + a_t = n, a_j \in \{1, 4\}} k_t(a_1, \dots, a_t) \geq (\sqrt{2})^n \quad (20)$$

(here minimum is taken over all t -tuples a_1, \dots, a_t such that $a_1 + \dots + a_t = n$ and $a_j \in \{1, 4\}$). This can be easily verified by induction in n . The base of induction for $n = 23, 24$ is checked by computer by MAPLE (the program is given in section 10). By the Sylvester theorem any natural number t greater than $505 = 23 \times 24 - 23 - 24$ can be expressed in the form $t = 23x + 24y$ with nonnegative integers x, y . Hence for $t \geq 506$ we have

$$k_t(a_1, \dots, a_t) \geq \prod_{1 \leq j \leq x} k_{23}(a_1^{(j)}, \dots, a_{23}^{(j)}) \prod_{1 \leq j \leq y} k_{24}(b_1^{(j)}, \dots, b_{24}^{(j)})$$

(here $(a_1, \dots, a_t) = (a_1^{(1)}, \dots, a_{23}^{(1)}, \dots, a_1^{(x)}, \dots, a_{23}^{(x)}, b_1^{(1)}, \dots, b_{24}^{(1)}, \dots, b_1^{(y)}, \dots, b_{24}^{(y)})$).

Now (20) follows from the base of induction for $n = 23, 24$. Theorem 3 is proved.

11. MAPLE program for verifying the inequalities for $n = 23, 24$. Here is the program for $n = 23$. The program for $n = 24$ is quite similar.

```

for a1 from 1 by 3 to 4 do
for a2 from 1 by 3 to 4 do
for a3 from 1 by 3 to 4 do
for a4 from 1 by 3 to 4 do
for a5 from 1 by 3 to 4 do
for a6 from 1 by 3 to 4 do
for a7 from 1 by 3 to 4 do
for a8 from 1 by 3 to 4 do
for a9 from 1 by 3 to 4 do
for a10 from 1 by 3 to 4 do
for a11 from 1 by 3 to 4 do
for a12 from 1 by 3 to 4 do
for a13 from 1 by 3 to 4 do
for a14 from 1 by 3 to 4 do
for a15 from 1 by 3 to 4 do
for a16 from 1 by 3 to 4 do
for a17 from 1 by 3 to 4 do
for a18 from 1 by 3 to 4 do
for a19 from 1 by 3 to 4 do
for a20 from 1 by 3 to 4 do
for a21 from 1 by 3 to 4 do

```


[illegible]

end do;
end do;
end do;
end do;

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